Graph-truncations of 3-polytopes.

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Abstract

In this paper we study the operation of cutting off edges of a simple 3-polytope P along the graph Γ. We give the criterion when the resulting polytope is simple and when it is flag. As a corollary we prove the analog of Eberhard's theorem about the realization of polygon vectors of simple 3-polytopes for flag polytopes.

1 Introduction.

For the introduction to the polytope theory we recommend the books [Gb03, Z07].

Definition 1.1. A *convex polytope* P is a set

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{a}_i\boldsymbol{x} + b_i \geqslant 0, i = 1, \dots, m\}$$

Let this representation be *irredundant*, that is deletion of any inequality changes the set. Then each hyperplane $\mathcal{H}_i = \{\boldsymbol{x} \in \mathbb{R}^n \colon \boldsymbol{a}_i\boldsymbol{x} + b_i = 0\}$ defines a *facet* $F_i = P \cap \mathcal{H}_i$.

In the following by a *polytope* we mean a convex polytope.

A *dimension* $\dim(P)$ of the polytope P is defined as $\dim \operatorname{aff}(P)$. We will consider n-dimensional polytopes (n-polytopes) in $\mathbb{R}^n$.

A *face* F of a polytope is an intersection $F = P \cap \{\boldsymbol{a}\boldsymbol{x} + b = 0\}$ for some *supporting hyperplane* $\{\boldsymbol{a}\boldsymbol{x} + b = 0\}$, i.e. $\boldsymbol{a}\boldsymbol{x} + b \geqslant 0$ for all $\boldsymbol{x} \in P$. Each face is a convex polytope itself. 0-dimensional faces are called *vertices*, 1-dimensional faces – *edges*, $(n-1)$-faces – *facets*. It can be shown that the set of all facets is $\{F_1, \dots, F_m\}$. Intersection of any set of faces of polytope is a face again (perhaps empty).

A vertex of an n-polytope P is called *simple* if it is contained in exactly n facets. An n-polytope P is called *simple*, if all it's vertices are simple. Each k-face of a simple polytope is an intersection of exactly $n-k$ facets.

A *combinatorial polytope* is an equivalence class of combinatorially equivalent convex polytopes, where two polytopes are *combinatorially equivalent* if there is an inclusion-preserving bijection of the sets of their faces.

A simple polytope is called *flag* if any set of pairwise intersecting facets $F_{i_1}, \dots, F_{i_k} \colon F_{i_s} \cap F_{i_t} \neq \varnothing$ has nonempty intersection $F_{i_1} \cap \dots \cap F_{i_k} \neq \varnothing$.

A *non-face* is the set $\{F_{i_1}, \dots, F_{i_k}\}$ with $F_{i_1} \cap \dots \cap F_{i_k} = \varnothing$. A *missing face* is an inclusion-minimal non-face.

The following results are well-known.

Proposition 1.2. *A polytope P is flag if and only if all its missing faces have cardinality 2.*

Proposition 1.3. *Each face of a flag polytope is a flag polytope again.*

Proposition 1.4. *The simplex Δ^n is not flag for $n \geqslant 3$. A 3-polytope $P^3 \neq \Delta^3$ is not flag if and only if it has missing face of cardinality 3: $\{F_i, F_j, F_k\}$, $F_i \cap F_j, F_j \cap F_k, F_k \cap F_i \neq \varnothing$, $F_i \cap F_j \cap F_k = \varnothing$.*

Definition 1.5. Missing face $\{F_i, F_j, F_k\}$ of a 3-polytope P^3 we will also call a 3-*belt*.

Let $f_i(P)$ be the number of i-faces of the polytope P.

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Proposition 1.6 (The Euler formula)**.** *For a 3-polytope we have*

$$f_0 - f_1 + f_2 = 2$$

Let p_k be the number of 2-faces of P that are k-gons.

Proposition 1.7. *For a simple 3-polytope P^3 we have*

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{k\geqslant 7}(k-6)p_k \qquad (*)$$

Proof. Let us count the number of pairs (edge, it's vertex). It is equal to $2f_1$, and since P is simple to $3f_0$. Then $f_0 = \frac{2f_1}{3}$ and from the Euler formula we obtain $2f_1 = 6f_2 - 12$. Then counting the pairs (facet, it's edge) we have

$$\sum_{k\geqslant 3} kp_k = 2f_1 = 6\left(\sum_{k\geqslant 3} p_k\right) - 12,$$

which implies the formula (*). ☐

Theorem 1.8 (Eberhard)**.** [Eb1891]
For every sequence $(p_k|3 \leqslant k \neq 6)$ of nonnegative integers satisfying (*), there exist values of p_6 such that there is a simple 3-polytope P^3 with $p_k = p_k(P^3)$ for all $k \geqslant 3$.

2 Graph-truncations

Construction 2.1. Consider a subgraph Γ without isolated vertices in the edge-vertex graph $G(P)$ of a simple 3-polytope P. For each edge $E_{i,j} = F_i \cap F_j = P \cap \{\boldsymbol{x} \in \mathbb{R}^3 \colon (\boldsymbol{a}_i + \boldsymbol{a}_j)\boldsymbol{x} + (b_i + b_j) = 0\}$ consider the halfspace $\mathcal{H}^+_{ij,\varepsilon} = \{\boldsymbol{x} \in \mathbb{R}^3 \colon (\boldsymbol{a}_i + \boldsymbol{a}_j)\boldsymbol{x} + (b_i + b_j) \geqslant \varepsilon\}$. Set

$$P_{\Gamma,\varepsilon} = P \cap \bigcap_{E_{i,j}\in\Gamma} \mathcal{H}^+_{ij,\varepsilon}$$

For small values of ε the combinatorial type of $P_{\Gamma,\varepsilon}$ does not depend on ε. We will denote it P_Γ and call a *graph-truncation* of P.

Facets of the polytope P_Γ are in one-to one correspondence with facets F_i of P (denote such facets by the same symbol F_i) and edges $F_i \cap F_j \in \Gamma$ (denote such facets as $F_{i,j}$).

Proposition 2.2. *The polytope P_Γ is simple if and only if the graph Γ does not contain vertices of valency 2.*

Proof. For small ε all new vertices of the polytope $P_{\Gamma,\varepsilon}$ lie in small neighborhoods of vertices of P. Consider a vertex $\boldsymbol{v} = F_{i_1} \cap F_{i_2} \cap F_{i_3}$ of P and introduce new coordinates in $\mathbb{R}^3$ by the formulas

$$y_1 = \boldsymbol{a}_{i_1}\boldsymbol{x} + b_{i_1}, \quad y_2 = \boldsymbol{a}_{i_2}\boldsymbol{x} + b_{i_2}, \quad y_3 = \boldsymbol{a}_{i_3}\boldsymbol{x} + b_{i_3}.$$

In new coordinates in some neighborhood $U(\boldsymbol{v})$ of $\boldsymbol{v} = \boldsymbol{0}$ the polytope $P_{\Gamma,\varepsilon}$ has irredundant representation

$$P_{\Gamma,\varepsilon} \cap U(\boldsymbol{v}) = \{\boldsymbol{y} \in \mathbb{R}^3 \colon y_1, y_2, y_3 \geqslant 0; y_p + y_q \geqslant \varepsilon, \text{ if } F_{i_p} \cap F_{i_q} \in \Gamma\}$$

The polytope $P_{\Gamma,\varepsilon}$ has non-simple vertex in $U(\boldsymbol{v})$ if and only if there is some point in $P_{\Gamma,\varepsilon} \cap U(\boldsymbol{v})$ that belongs to 4 facets.

If $\boldsymbol{v} \notin \Gamma$ then $\boldsymbol{v} \in P_{\Gamma,\varepsilon}$ is the only vertex in $U(\boldsymbol{v})$ and it is simple.

If $\boldsymbol{v}$ has valency 1 in Γ, say $F_{i_1} \cap F_{i_2} \in \Gamma$, and some point lies in $F_{i_1}, F_{i_2}, F_{i_3}, F_{i_1,i_2}$, then $y_1 = y_2 = y_2 = 0$ and $y_1 + y_2 = \varepsilon$. Contradiction.

If $\boldsymbol{v}$ has valency 2 in Γ, say $F_{i_1} \cap F_{i_2}, F_{i_2} \cap F_{i_3} \in \Gamma$, then the point $(0, \varepsilon, 0) \in P_{\Gamma,\varepsilon}$ belongs to $F_{i_1}, F_{i_3}, F_{i_1,i_2}, F_{i_2,i_3}$, so $P_{\Gamma,\varepsilon}$ is not simple.

Let $\boldsymbol{v}$ has valency 3 in Γ and some point belongs to 4 facets. If there are F_{i_1,i_2}, F_{i_2,i_3} and F_{i_3,i_1}, among them, then $y_1 = y_2 = y_3 = \frac{\varepsilon}{2}$, and there can not be neither F_{i_1}, nor F_{i_2}, nor F_{i_3}. Therefore there should be at least two of $F_{i_1}, F_{i_2}, F_{i_3}$, say F_{i_1}, F_{i_2}. Then $y_1 = y_2 = 0$. But $y_1 + y_2 \geqslant \varepsilon$. Contradiction. So $P_{\Gamma,\varepsilon}$ has only simple vertices in $U(\boldsymbol{v})$ in this case. ☐

Theorem 2.3. *A simple 3-polytope P_Γ is flag if and only if any triangular facet of P contains no more than one edge in Γ and for any 3-belt (F_i, F_j, F_k) of P one of the edges $F_i \cap F_j$, $F_j \cap F_k$, $F_k \cap F_i$ belongs to Γ.*

Proof. Since P_Γ is simple, Proposition 2.2 implies that valency of each vertex of Γ is 1 or 3.

If P contains a 3-belt (F_i, F_j, F_k), such that $F_i \cap F_j$, $F_j \cap F_k$, $F_k \cap F_i \notin \Gamma$, then (F_i, F_j, F_k) is either a 3-belt in $P_{\Gamma,\varepsilon}$. Consider a triangular face of P. If exactly two it's edges belong to Γ, then Proposition 2.2 implies that valency of their common vertex is 3 and other vertices have valency 1 in Γ. If all tree edges belong to Γ, then all their vertices have valency 3 in Γ. In both cases after truncation the face remains to be triangular, so $P_{\Gamma,\varepsilon}$ in not flag. Thus we proved the only if part of the theorem .

$P_{\Gamma,\varepsilon} \neq \Delta^3$, since it contains more than 4 facets. Therefore if it is not flag, then there is a 3-belt (G_1, G_2, G_3) in $P_{\Gamma,\varepsilon}$ by Proposition 1.4.

If $G_1 = F_i$, $G_2 = F_j$, $G_3 = F_k$, then either (F_i, F_j, F_k) is a 3-belt in P, or there is a vertex $\boldsymbol{v} = F_i \cap F_j \cap F_k \in P$. In the first case one of the edges $F_i \cap F_j$, $F_j \cap F_k$, or $F_k \cap F_i$ belongs to Γ and is cut off when we pass to $P_{\Gamma,\varepsilon}$, so the corresponding facets do not intersect in $P_{\Gamma,\varepsilon}$. In the second case the vertex $\boldsymbol{v}$ is cut off, since $F_i \cap F_j \cap F_k = \varnothing$ in $P_{\Gamma,\varepsilon}$. It is possible only if we cut off one of the edges containing this vertex, so the corresponding two facets do not intersect in $P_{\Gamma,\varepsilon}$.

If $G_1 = F_i$, $G_2 = F_j$, and $G_3 = F_{p,q}$ correspond to an edge $E_{p,q} = F_p \cap F_q$ of P, then $F_i \cap F_j \neq \varnothing$ and $E_{p,q}$ intersects both F_i and F_j. Since $F_i \cap F_j$ was not cut off, we have $\{i, j\} \neq \{p, q\}$. If $i \in \{p, q\}$ or $j \in \{p, q\}$, then the edge $F_p \cap F_q$ intersects the edge $F_i \cap F_j$ at the vertex, so the facets F_i, F_j, and $F_{p,q}$ have common vertex in $P_{\Gamma,\varepsilon}$. Now let $\{i, j\} \cap \{p, q\} = \varnothing$. Then (F_i, F_j, F_p) or (F_i, F_j, F_q) is a 3-belt in P. Otherwise $F_i \cap F_j \cap F_p \neq \varnothing$, $F_i \cap F_j \cap F_q \neq \varnothing$, $F_p \cap F_q \cap F_i \neq \varnothing$, and $F_p \cap F_q \cap F_j \neq \varnothing$, therefore $P = \Delta^3$, all it's facets are triangles and in any triangle no more than one edge is cut off. Since $F_i \cap F_j \notin \Gamma$ and $F_p \cap F_q \in \Gamma$, in facets F_p and F_q the only edge $F_p \cap F_q$ is cut off and Γ contains no other edges. Then the facets F_p and F_q are triangles in $P_{\Gamma,\varepsilon}$ either, and it is not flag. By assumption one of the edges of the 3-belt we obtain belongs to Γ. Since the edge $F_i \cap F_j$ was not cut off, one of the edges $F_i \cap F_p$, $F_j \cap F_p$, $F_i \cap F_q$ and $F_j \cap F_q$ belongs to Γ and was cut off, say $F_i \cap F_p$. Then $F_i \cap F_{p,q} = \varnothing$, which is a contadiction.

If only one of the facets (G_1, G_2, G_3) corresponds to a facet of P, say $G_1 = F_i$, then two other facets correspond to edges of P that both intersect F_i and have common vertex. If both edges belong to F_i, then $F_i \cap G_2 \cap G_3 \neq \varnothing$. If exactly one of them belong to F_i, say corresponding to G_2, then $F_i \cap G_3 = \varnothing$. At last, if both of them do not belong to F_i, then their common vertex $\boldsymbol{v}$ do not belong to F_i, and these two edges and their common vertex define some facet F_j that has with F_i two common vertices – the remaining ends of two edges, thus $F_i \cap F_j$ is an edge, connecting these vertices. Then F_j is a triangle containing two edges in Γ. Contradiction.

At last if all three facets of 3-belt correspond to edges of P, then these edges pairwise intersect. Two of them define some facet F_i. If the third edge does not belong to F_i, then all three edges have common vertex and in $P_{\Gamma,\varepsilon}$ the corresponding facets have a common vertex either. If the third edge belongs to F_i, then F_i is a triangle with three edges in Γ. Contradiction.

Thus we have considered all possible cases, and the theorem is proved. ◻

3 Application

As an application of Theorem 2.3 we prove an analog of Eberhard's theorem for flag 3-polytopes. Since any face of a flag polytope is flag itself, we have $p_3(P^3) = 0$ for any flag polytope.

Theorem 3.1. *For every sequence $(p_k | 3 \leqslant k \neq 6)$ of nonnegative integers satisfying $p_3 = 0$ and (*), there exist values of p_6 such that there is a flag simple 3-polytope P^3 with $p_k = p_k(P^3)$ for all $k \geqslant 3$.*

Proof. From Eberhard's Theorem 1.8 it follows that there exist values of p_6 such that there is a polytope P^3 with $p_k = p_k(P)$ for all $k \geqslant 3$. Let us consider the graph $\Gamma = G(P^3)$. Since $p_3 = 0$, we obtain from Theorem 2.3 that $P_{G(P)}$ is flag. On the other hand, facets of $P_{G(P)}$ are in one-to-one correspondence with facets and edges of P. Moreover, k-gonal facets of P correspond to k-gonal facets of $P_{G(P)}$ and edges of P correspond to 6-gonal facets of $P_{G(P)}$. Therefore

$$p_k(P_{G(P)}) = \begin{cases} p_k(P), & k \neq 6; \\ p_6(P) + f_1(P), & k = 6. \end{cases}$$

This proves the theorem. $\square$

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